

Analytic, quasineutral, two-dimensional Maxwell-Vlasov equilibria

G. N. Throumoulopoulos¹, H. Tasso²

¹*University of Ioannina, Association Euratom - Hellenic Republic,
Section of Theoretical Physics, GR 451 10 Ioannina, Greece*

²*Max-Planck-Institut für Plasmaphysik, Euratom Association,
D-85748 Garching, Germany*

Abstract

Two-dimensional Maxwell-Vlasov equilibria with finite electric fields, axial (“toroidal”) plasma flow and isotropic pressure are constructed in plane geometry by using the quasineutrality condition to express the electrostatic potential in terms of the vector potential. Then for Harris-type distribution functions, Ampere’s equation becomes of Liouville type and can be solved analytically. As an example, a periodic “cat-eyes” steady state consisting of a row of magnetic islands is presented. The method can be extended to (toroidal) axisymmetric equilibria.

Equilibrium is the starting point for stability and transport studies of astrophysical and laboratory plasmas. In the framework of collisionless kinetic theory, equilibrium states should be constructed as self consistent solutions of Vlasov and Maxwell equations. To this end, the knowledge of constants of motion for the particles in the continuum approximation (microfluids) is of crucial importance because then the general solution of Vlasov equation can be written as an arbitrary function of the complete set of constants of motion. This is feasible only for one-dimensional equilibria, i.e. in this case the three constants of motion are the energy ($H_s = (1/2)m_s v^2 + q_s \Phi(y)$) and the two canonical momenta ($p_{xs} = m_s v_x + q_s A_x(y)$ and $p_{zs} = m_s v_z + q_s A_z(y)$); consequently, the distribution functions are of the form $f_s = f_s(H_s, p_{xs}, p_{zs})$. Here, (x, y, z) are Cartesian coordinates, $v^2 = v_x^2 + v_y^2 + v_z^2$, $\Phi(y)$ the electrostatic potential, $A_x(y)$ and $A_z(y)$ the components of the vector potential, and the subscript s denotes the particle species. Unlikely, for two-dimensional equilibria the complete set of constants of motion is missing, i.e. only the energy H_s and the momentum $p_{zs} = m_s v_z + q_s A_z(x, y)$ conjugate to the ignorable coordinate z are known out of the four constants of motion. A good number of solutions for one-dimensional [1]-[7] and two-dimensional [8]-[10] equilibria were constructed on the basis of modified Maxwellian distribution functions of the forms $f_s = \exp(-\beta_s H_s) g_s(p_{xs}, p_{zs})$ and $f_s = \exp(-\beta_s H_s) g_s(p_{zs})$, respectively, with g_s arbitrary functions of the conserved momenta and $\beta_s = 1/(k_B T_s)$. These equilibria concern neutral plasmas in connection with a special set of distribution functions such that it holds (for one-dimensional equilibria) $N_i(A_x, A_z) = N_e(A_x, A_z)$, where N_i (N_e) is the ion (electron) density [1]; viz. in addition to the usual quasineutrality condition it was assumed that $N_i(A_x, A_z)$ is the same function of A_x and A_z as $N_e(A_x, A_z)$, thus leading to a vanishing electrostatic potential. On physical grounds this additional assumption is oversimplifying because it ignores the mass difference of ions and electrons. Also, finite electric fields associated with macroscopic plasma (ion) flows are important in laboratory fusion plasmas for the transitions from low to high confinement modes of operation.

Aim of the present study is to construct analytically a class of quasineutral, two dimensional Maxwell-Vlasov equilibria. This is accomplished by employing the quasineutrality condition (without the additional assumption of functionally identical ion and electron densities) to express Φ as a function of $A_z(x, y)$. A similar method was employed by Mynick and coauthors [11] to construct numerically by an iteration algorithm one-dimensional quasineutral equilibria. Also, the method was reviewed recently in Sec. II of Ref.

[12].

We consider a plasma of electrons and protons at equilibrium with a current density in the axial (“toroidal”) z -direction. Consequently, the vector potential has a single component $A_z(x, y)$. In addition to the “poloidal” magnetic field with components $B_x(x, y)$ and $B_y(x, y)$ associated with A_z , we include for stabilizing reasons a constant axial magnetic field B_{z0} which otherwise does not affect the equilibrium. Furthermore, we employ Harris-type distribution functions,

$$f_s(H_s, p_{zs}) = n_{0s} \left(\frac{m_s \beta_s}{2\pi} \right)^{3/2} \exp(-\beta_s H_s) \exp(\beta_s V_{zs} p_{zs}), \quad (1)$$

where V_{zs} are constant average (fluid) velocities and n_{0s} reference densities corresponding to Maxwellian distribution functions ($V_{zs} = 0$). Using the quasineutrality condition, $N_i = N_e$, where

$$N_s = n_{0s} \left(\frac{m_s \beta_s}{2\pi} \right)^{3/2} \exp(-q_s \beta_s \Phi) \int_{-\infty}^{\infty} \exp(-\beta_s m_s v^2 / 2) \exp(\beta_s V_{zs} p_{zs}) d^3v$$

(with $q_i = e$ and $q_e = -e$), the electrostatic potential can be expressed in terms of $A_z(x, y)$ as

$$\Phi(A_z) = \log \left[\frac{n_{0i}}{n_{0e}} \exp \left(A_z e V_{ze} \beta_e - \frac{1}{2} m_e V_{ze}^2 \beta_e + A_z e V_{zi} \beta_i + \frac{1}{2} m_i V_{zi}^2 \beta_i \right) \right]^{\frac{1}{e(\beta_e + \beta_i)}}. \quad (2)$$

Note that for (1) all the integrations of interest in the velocity space can be performed analytically. Using (1) and (2) one finds for the current density ($j_z = \sum_s q_s \int_{-\infty}^{\infty} v_z f_s d^3v$):

$$j_z(A_z) = e n_{0e}^{\frac{\beta_i}{\beta_e + \beta_i}} n_{0i}^{1 - \frac{\beta_i}{\beta_e + \beta_i}} (V_{zi} - V_{ze}) \exp \left[\frac{1}{2} V_{zi} (2A_z e + m_i V_{zi}) \beta_i - \frac{\beta_i \left(A_z e V_{ze} \beta_e - \frac{1}{2} m_e V_{ze}^2 \beta_e + A_z e V_{zi} \beta_i + \frac{1}{2} m_i V_{zi}^2 \beta_i \right)}{\beta_e + \beta_i} \right]. \quad (3)$$

Therefore, Ampere’s equation ($\nabla^2 A_z(x, y) = -\mu_0 j_z(A_z)$) assumes a Liouville-type form

$$\nabla^2 A_z = a \exp(b A_z) \quad (4)$$

where

$$a = \mu_0 (V_{ze} - V_{zi}) e n_{0e}^{\frac{\beta_i}{\beta_e + \beta_i}} n_{0i}^{\frac{\beta_e}{\beta_e + \beta_i}} \exp \left[\frac{(m_e V_{ze}^2 + m_i V_{zi}^2) \beta_i \beta_e}{2 (\beta_e + \beta_i)} \right], \quad (5)$$

$$b = \frac{e (V_{zi} - V_{ze}) \beta_e \beta_i}{\beta_e + \beta_i}. \quad (6)$$

The equilibrium for $V_{zi} = 0$ is static, viz. only the electrons are in non-thermal motion to produce the axial current. For $V_{zi} \neq 0$, however, there is a constant ion-fluid axial velocity, $\int_{-\infty}^{\infty} v_z f_i d^3v / N_i = V_{zi}$, and the ion motion contributes to j_z . It may be noted here that for distribution functions of the form $f_s(H_s, p_{zs})$ it is not possible to create poloidal plasma velocities because of the two missing constants of motion. Even the third constant of motion found in Ref. [13] near the magnetic axis does not help to this end because poloidal flows vanish on axis. For the equilibrium constructed here the pressure is isotropic, i.e. the pressure tensor,

$$P_{ij} = \sum_s m_z \int_{-\infty}^{\infty} (v_i - \langle v_i \rangle_s) (v_j - \langle v_j \rangle_s) f_s d^3v, \quad i, j = x, y, z,$$

is diagonal with $P_{xx} = P_{yy} = P_{zz} \equiv P$ (see also Fig. 3). For $V_{zi} = V_{ze}$ it follows that the current density vanishes ($a = b = 0$) and (4) reduces to Laplace equation. Therefore, A_z can not be constant on any closed curve in the (x, y) plane without being constant in the region within this curve. Consequently, the electrostatic potential Φ is constant too in this region because of (2) and the distribution functions become spatially uniform; hence, one recovers the well-known equilibrium solution of the Maxwell-Vlasov equations for which all quantities are homogeneous. No “confined solution” is possible in this case. Also, it is noted here that for $V_{zi} = V_{ze}$ solution (8) below, though pertinent to an unbounded plasma, becomes singular ($\tilde{A}_z \rightarrow \infty$).

Introducing dimensionless quantities ($\xi = x/L$, $\eta = y/L$, $\tilde{A}_z = A_z/(B_{z0}L)$, $\tilde{a} = aL/B_{z0}$, $\tilde{b} = bB_{z0}L$ with L a length scaling parameter), the general solution of (4) is given by [14, 15]

$$\begin{aligned} \tilde{A}_z(x, y) &= \frac{\chi |\tilde{a}\tilde{b}| - \tilde{a}\tilde{b} \log |\tilde{a}\tilde{b}|}{\tilde{a}\tilde{b}^2}, \\ \chi &= \log \left[\frac{(u^2 + v^2 + 1)^2}{8 (u_\xi^2 + u_\eta^2)} \right], \end{aligned} \quad (7)$$

where $u(\xi, \eta)$ and $v(\xi, \eta)$ are real conjugate functions resulting from $w(\xi + i\eta) = u + iv$, with w a differentiable arbitrary generating function. As an example of complete equilibrium construction we consider here the function

$$w(\zeta) = \sqrt{\frac{1+\epsilon}{1-\epsilon}} \tan\left(\frac{\zeta}{2}\right)$$

with $\zeta = \xi + i\eta$, and ϵ a parameter such that $|\epsilon| \leq 1$. For this choice of w , (7) acquires the form

$$\tilde{A}_z(\xi, \eta) = \log \left\{ \frac{2(1-\epsilon^2)}{|\tilde{a}\tilde{b}| [\cosh(\xi) - \epsilon \cos(\eta)]^2} \right\}^{1/\tilde{b}}. \quad (8)$$

The function \tilde{A}_z labels the magnetic surfaces. The equilibrium configuration shown in Fig. 1 consists of an infinite row of identical periodic islands known as “cat-eyes” (see for example Ref. [16]). The islands have magnetic axes at $\xi_a = 2k\pi$, $\eta_a = 0$ and a separatrix with x -points at $\xi_x = (2k+1)\pi$, $\eta_x = 0$ where k an integer. The ordinates of the separatrix are located at $\xi = \xi_a$, $\eta = \eta_s = \pm \operatorname{arctanh}(1+2\epsilon)$ (see Fig. 1). The equilibrium has the following free parameters: n_{0s} , β_s , V_{zs} ($s = e, i$), B_{z0} , ϵ , and L . The dependent parameters a and b (Eqs. (5 and (6)) relate features of the distribution function to macroscopic equilibrium characteristics (Eq. (8)). For $V_{zi} \neq V_{ze}$ the physical quantities (\mathbf{B} , $\mathbf{E} = -(d\Phi/dA_z)\nabla A_z$, j_z and P) are everywhere regular and vanish as y tends to infinity except for E_y which in this limit approaches a finite value. Profiles of E_x and E_y are shown in Fig. 2 for the following fusion relevant values of the free parameters: $n_{0i} = n_{0e} = 10^{19}m^{-3}$, $k_B T_i = k_B T_e = 1keV$, $V_{zi} = 10^4 m/sec$, $V_{ze} = 10V_{zi}$, $B_{z0} = 1T$, $L = 1m$, and $\epsilon = 0.6$. Also, the y -profiles of j_z and P have an extremum on the magnetic axis (Fig. 3). For $\epsilon = 0$ the configuration becomes one-dimensional; this is an extension of the Harris sheet equilibrium [17] (usually employed as initial state in reconnection studies) with finite \mathbf{E} and constant axial velocity.

Quasineutral equilibria with sheared axial flow which may be more pertinent to the improved confinement modes can be constructed by the alternative distribution functions

$$f_s(H_s, p_{zs}) = n_{0s} \left(\frac{m_s \beta_s}{2\pi} \right)^{3/2} \exp(-\beta_z H_s) \exp\left(\frac{\beta_{zs} p_{zs}^2}{2m_s}\right), \quad (9)$$

with β_s and β_{zs} constants. A similar procedure then leads to A_z -dependent average axial velocities:

$$\frac{\int_{-\infty}^{\infty} v_z f_s d^3v}{N_s} = \frac{q_s \beta_{zs}}{m_s (\beta_s - \beta_{zs})} A_z(x, y),$$

and Ampere's equation assumes the form

$$\nabla^2 A_z = a_1 A_z \exp(b_1 A_z^2), \quad (10)$$

where the parameters a_1 and b_1 are known functions of n_{0s} , β_s and β_{zs} . Eq. (10), higher nonlinear than (4), should be solved numerically.

In summary, using the quasineutrality condition to express the electrostatic potential in terms of the vector potential and Harris-type distribution functions (Eq. (1)) we have constructed a class of plane, two-dimensional Maxwell-Vlasov equilibria with finite electric fields, constant axial plasma velocity and isotropic pressure. The equilibrium was exemplified by the cat-eyes solution. Equilibria with sheared axial flow can be derived by alternative distribution functions, e.g. (9). The method can also be applied in (toroidal) axisymmetric and helically symmetric geometries.

One of the authors (GNT) would like to thank Prof. H. Weitzner for useful discussions. Part of this work was conducted during a visit of GNT to the Max-Planck-Institut für Plasmaphysik, Garching. The hospitality of that Institute is greatly appreciated. This work was performed within the participation of the University of Ioannina in the Association Euratom-Hellenic Republic, which is supported in part by the European Union and by the General Secretariat of Research and Technology of Greece. The views and opinions expressed herein do not necessarily reflect those of the European Commission.

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Figure captions

Fig. 1: \tilde{A}_z -lines of the cat-eyes solution (7) as intersections of the magnetic surfaces with the poloidal plane.

Fig. 2: Profiles of the electric field components E_x and E_y associated with the cat-eyes solution (7). The profiles $E_x(y)$ and $E_y(x)$ have been chosen at $x/L = \pi/2$ and $y/L = 0.5$, respectively, because $E_x(x=0, y) = E_y(x, y=0) \equiv 0$.

Fig. 3: y -profiles at $x = 0$ of the current density, j_z , and the pressure, P , associated with the cat-eyes solution (7).

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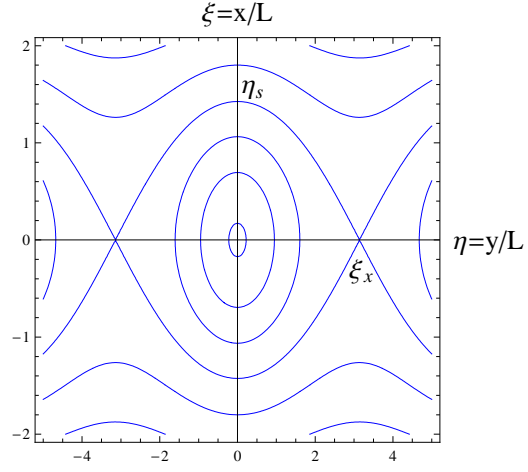


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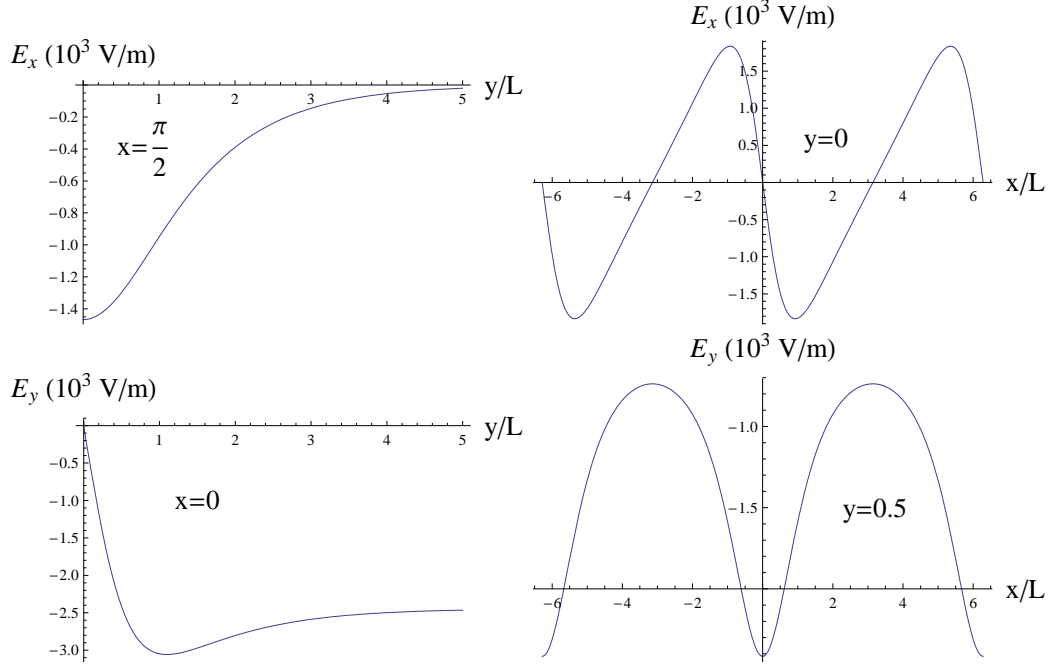


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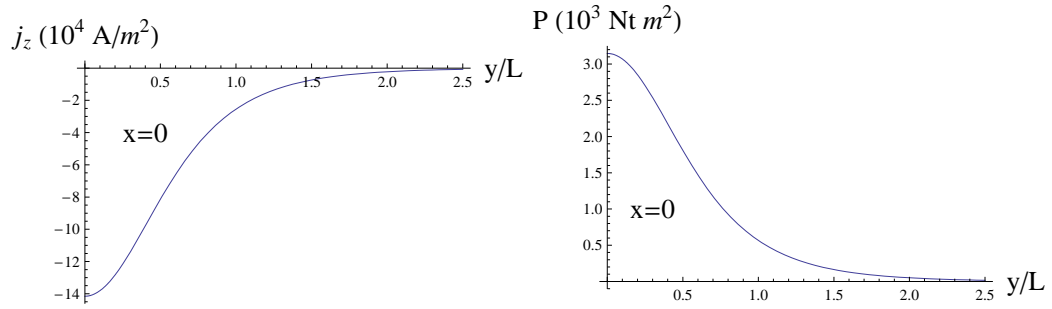


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